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LABELLING SYSTEMS AND R.E. STRUCTURES

C.J. ASH

Department of Mathematics, Monash University, Wellington Road, Clayton, Victoria 3168, Australia

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We give a new, simpler, yet slightly more general presentation of the labelling systems of [2]. These were introduced in [1, 2] in order to treat various topics in the study of recursive structures, as was done in [2, 3, 5, 6]. The new systems may be used wherever the old ones were, and more simply, and they may also be used in constructing r.e. structures, rather than just recursive ones, whereas, it seems, the old ones cannot. We anticipate that they will find still wider applications.

The object of the present paper is to describe the new systems, to establish their significance in Proposition 1, which applies equally to limit ordinals, and to prove, in Lemmas 1 to 5, the relevant results concerning the recursive infinitary syntax of r.e. structures. We content ourselves with a single example, in the final section, of a theorem in which these systems can be used to construct r.e. structures rather than only recursive structures.

This example seems quite sufficient both to show from first principles how the systems may be used and also to illustrate how the results of [2, 3, 5, 6] may be modified or generalized in the case of r.e. structures. We also show, using the same example, how these systems can equally well be applied when one wishes to consider only recursive structures.

In Section 1, we make some introductory remarks concerning recursive and r.e. structures and their study.

In Section 2, we describe our new α-systems and state the result, Proposition 1, which establishes their significance. We add some comments on the reasons for our choice of conventions.

For a proof of Proposition 1 when α is infinite, we would be unable to do better than to reproduce almost word-for-word the rather long and intricate argument of [2], so in Section 3 we adopt what seem to be a preferable course and assume, for that section only, complete familiarity with the arguments of [2]. We describe how these arguments can be modified to establish Proposition 1 for successor ordinals and a different version, Proposition 2, previously described in Section 2, for limit ordinals.

Presumably, Proposition 1 for limit ordinals could also be obtained by judicious changes to the argument of [2], but we have noticed that it can instead be deduced directly from Proposition 2, by constructing one α-system from another, and we give this construction in Section 4.

For applications to r.e. structures, we need to consider suitable classes of recursive infinitary formulas, slightly different from those of [2], and in Section 5 we define these and some related notions. In Lemmas 1 to 5 we establish the properties of these which are need for Section 6 and which may also be useful elsewhere.

In Section 6 we establish, under certain assumptions, necessary and sufficient condition on structures $\mathfrak{A}$ and $\mathfrak{B}$ and ordinals $\alpha < \omega_1^{\mathrm{CK}}$ that an arbitrary Π_α^0 set S can be encoded in a recursive sequence $\langle \mathfrak{C}_n \rangle$ of r.e. structures, in the sense that, for each n, we have

$$\mathfrak{C}_n \cong \begin{cases} \mathfrak{A} & \text{if } n \in S, \\ \mathfrak{B} & \text{if } n \notin S. \end{cases}$$

We observe that this is not only an analogue of the result of [5] for recursive structures, but also a generalization of it. For the same reason, Lemmas 1 to 5 are both analogues and generalizations of various results in [2, 3, 5].

Apart from Section 3, the paper is fairly self-contained, and the reader may wish to omit Section 3 and accept Proposition 1 on trust. In this case, on first reading, he may prefer also to omit the further remarks on limit α-systems in Section 2 and the construction which constitutes Section 4.

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0. Preliminaries

Formulas

The formulas considered are the infinitary formulas of the appropriate $L_{\omega_1\omega}$ as defined in [7]. We use the term *positive* formula in a slightly unconventional sense to means a formula containing no negation symbols *except in the negated atomic formulas* $x_i \neq x_j$. They therefore would be the positive formulas in the conventional sense if a new symbol for $\neq$ were introduced. The use of the sign $+$ in the natation Σ_α^+ and Π_α^+ for classes of formulas *definitely does not* mean that the formulas are positive, as explained in Section 5.

Sequences

We use the notation $\bar{a}$ to denote a configuration of the form $a_1, a_2, \ldots, a_n$ which may, depending on the context, also be construed as the set $\{a_1, a_2, \ldots, a_n\}$ or as the sequence $(a_1, a_2, \ldots, a_n)$. Thus $\bar{a}$, $\bar{b}$ may be used to denote the concatenation of the sequences $\bar{a}$ and $\bar{b}$. We also adopt the convention that when f is a function of the appropriate type, if $\bar{a}$ denotes $a_1, a_2, \ldots, a_n$, then $f(\bar{a})$ denotes $f(a_1), f(a_2), \ldots, f(a_n)$.

Ordinals

The symbols α, β, γ, ξ, η are used exclusively to denote ordinal numbers, while we use the notation $\mathbb{N}$ for the set of all natural numbers. The recursive ordinals and Kleene's system $\mathcal{O}$ of notations for them are described in [8]. In this paper all ordinals considered may be taken to be recursive ordinals. At any one time, all ordinals under consideration are $\beta \leq \alpha$ for some $\alpha < \omega_1^{\mathrm{CK}}$. When the terminology of recursive functions is applied to these ordinals β, we assume that some notation, a, for α has been chosen and that each $\beta < \alpha$ is treated according to its unique notation, $b \in \mathcal{O}$, for which $b <_{\mathcal{O}} a$. In the interest of readability we ask the reader to understand 'notation for β' in this sense, instead of 'β' when necessary.

1. R.e. structures

We define an *r.e. structure* to be one of the form $\mathfrak{A} = (A, \langle R_i \rangle_{i \in I}, \langle g_j \rangle_{j \in J}, \langle a_k \rangle_{k \in K})$ for which A, I, J, K are recursive sets and, for suitable recursive functions μ, ν, each R_i is uniformly in i an r.e. $\mu(i)$-ary recursive relation on A, each g_j is uniformly in j a $\nu(j)$-ary recursive function on A and the function $k \mapsto a_k$ from K to A is recursive. The *similarity type* of such a structure is (I, J, K, μ, ν) and the corresponding effective language is denoted by L.

The usual device for treating the g_j and the a_k as relations may be employed here without affecting our results, since a total function is recursive if and only if it has an r.e. graph. It may seem plausible to consider also the possibility of allowing A to be r.e. and not necessarily recursive, but one could then consider the inverse image of A under a one–one total recursive function and study instead the induced r.e. structure of the sort already described. For this reason, and since we have no non-trivial results in the case where A is finite, we may if we wish always assume that $A = \mathbb{N}$.

Several aspects of *recursive* structures have been considered in [1, 2, 3, 5, 6]. For recursive structures it is again sufficient to consider only structures of the form $(A, \langle R_i \rangle_{i \in I})$ where the R_i are now required to be uniformly *recursive* and not just r.e. We note that the study of r.e. structures includes the study of recursive structures, since recursive structures $(A, \langle R_i \rangle_{i \in I})$ correspond exact to r.e. structures $(A, \langle R_i \rangle_{i \in I}, \langle \bar{R}_i \rangle_{i \in I})$ where $\bar{R}_i$ denotes the complement of R_i. By the same device, it also includes the study of mixed structures $(A, \langle R_i \rangle_{i \in I}, \langle S_j \rangle_{j \in J}, \langle T_k \rangle_{k \in K})$ in which the R_i are required to be uniformly recursive, the S_j uniformly r.e. and the T_k uniformly co-r.e.

It is clear that a non-recursive permutation, f, of the set A in an r.e. structure $\mathfrak{A} = (A, \langle R_i \rangle_{i \in I})$ may still yield an r.e. structure, but having different recursive properties. It seems most elegant to view such a permutation as an isomorphism $f : \mathfrak{A} \cong \mathfrak{B}$ between r.e. structures, where $\mathfrak{B} = (A, \langle f(R_i) \rangle_{i \in I})$.

The topics considered in [1, 2, 3, 6] were all related, in one way or another, to the construction of such isomorphic recursive copies having various properties, which necessitated the construction, simultaneously, of Δ^0_α isomorphism for arbitrarily large $\alpha < \omega_1^{\mathrm{CK}}$. In [4], copies of a structure $\mathfrak{A}$ which are recursive *relative to a given Turing degree,* **d**, were considered, the problem being to determine from $\mathfrak{A}$ the least β for which there exists a least βth jump of such degrees **d**.

In this last connection one needs constructions which encode a given set in the βth jump of a structure but in no smaller jump. Such constructions suggested the topic of [5] in which the problem was to construct a recursive *sequence,* $\mathfrak{C}_0, \mathfrak{C}_1, \ldots$ of recursive structures for which the isomorphism type of $\mathfrak{C}_n$ depends, in a non-recursive way, on n. An example is that for any $\alpha < \omega_1^{\mathrm{CK}}$ and any $\Pi^0_{2\alpha+1}$ set S there exists such a recursive sequence for which

$$\mathfrak{C}_n \cong \begin{cases} \omega^\alpha & \text{if } n \in S, \\ \omega^{\alpha+1} & \text{if } n \notin S. \end{cases}$$

In Theorem 3.1 of [5] it was shown how such a recursive sequence can in general be constructed, subject to certain infinitary syntactical conditions on the structures involved. Since it seems to be the 'tidiest' use of the old labelling systems, we choose this result to generalize, in Section 6, to r.e. structures, thereby illustrating the use of the new systems.

2. Labelling systems

Definitions. An *r.e. scheme* is a structure $\mathscr{G} = (U, L, P, E)$ for which U, L are r.e. sets, P is an r.e. set of finite sequences of the form $(u_0, l_0, u_1, l_1, \ldots)$ where each $u_i \in U$ and each $l_i \in L$, and $E \subseteq L \times \mathbb{N}$ is r.e. For simplicity, we also require that P is closed under the formation of initial segments.

An *instruction* for $\mathscr{G}$ is a function p which assigns, to each member of P of the form $(u_0, l_0, \ldots, u_n, l_n)$, where $n \geq 0$, an element $u_{n+1} \in U$ for which $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}) \in P$.

An *input* for $\mathscr{G}$ is a triple (u, l, p) for which p is an instruction for $\mathscr{G}$ and $(u, l) \in P$. We say that (u, l, p) is Δ^0_α is p is.

A *run* in $\mathscr{G}$ of the input (u, l, p) is an infinite sequence $s = (u_0, l_0, u_1, l_1, \ldots)$ of which every finite initial segment is in P, such that $u_0 = u$, $l_0 = l$ and such that, for each $n \geq 0$, $u_{n+1} = p(u_0, l_0, \ldots, u_n, l_n)$.

For $l \in L$ we write $E(l) = \{m : (l, m) \in E\}$ and if $s = (u_0, l_0, u_1, l_1, \ldots)$ we write $E(s) = \bigcup_n E(l_n)$. We say that the run s in $\mathscr{G}$ is *conservative* if $E(s)$ is r.e.

Comment. Such a scheme corresponds to an infinite two-player game, where, alternately, player I plays elements of U and player II plays elements of L, so that at each stage the corresponding finite sequence remains in P.

An input corresponds to a starting position and a strategy for player I. Player II is considered to win against this strategy if the infinite sequence $l_0, l_1, \ldots$ of his plays makes $\bigcup_n E(l_n)$ r.e.

The name 'labelling system' arises from the following visualization of this situation. One pictures only the tree of positions after player I has acted, that is, the tree of sequences from P of the form $(u_0, l_0, \ldots, u_n, l_n, u_{n+1})$, and regards the action of player II not as proceeding to a successor node but rather as assigning a 'label' to the present node, consisting of an element of L. In this way one may picture the sequence of elements of U as nodes of a tree and use letters for the associated elements of L, as in [1], thereby avoiding the nuisance of remembering in the full tree which levels correspond to which player.

In applications, one designs an r.e. scheme in which the existence of a conservative run for a suitable input establishes the desired result. In terms of the game, one therefore wishes to ensure that player II may win when player I follows a certain strategy. In our applications so far, this strategy for player I consists invariably of supplying suitable information from a Δ^0_α oracle, for some fixed $\alpha < \omega_1^{\mathrm{CK}}$. In the cases where sharp results are established, there is no question but that player I has a winning strategy, which may be chosen to be $\Delta^0_{\alpha+1}$. Our desired result will follow if on the other hand we can prove that, for the scheme we have designed, player II can win against an arbitrary strategy for player I which is only Δ^0_α, because then, in particular, he can win against the one determined by the Δ^0_α oracle.

Proposition 1, to follow, asserts that the existence of r.e. relations on L satisfying certain conditions is sufficient to ensure this property of the scheme, for the value of α in question.

Definitions. An *α-system* where $1 \leqslant \alpha < \omega_1^{\mathrm{CK}}$ is a structure $\mathcal{S} = (U, L, P, E, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ for which (U, L, P, E) is an r.e. scheme, each $\subseteq_\xi$ is, uniformly in $\xi < \alpha$, an r.e. binary relation on L and the following conditions are satisfied.

(1) Each $\subseteq_\xi$ is reflexive and transitive.

(2) If $\xi < \eta < \alpha$ and $l \subseteq_\eta m$, then $l \subseteq_\xi m$.

(3) If $l \subseteq_0 m$ then $E(l) \subseteq E(m)$.

(4) If $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}) \in P$, if $\alpha > \xi_1 > \xi_2 > \cdots > \xi_k \geqslant 0$ and if $l_n = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \cdots \subseteq_{\xi_{k-1}} m_k$ then there exists l_{n+1} for which $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}, l_{n+1}) \in P$ and $m_i \subseteq_{\xi_i} l_{n+1}$ for each $i = 1, 2, \ldots, k$.

Proposition 1. *Every Δ^0_α input for any α-system has a conservative run.*

Moreover, there is a partial effective procedure which assigns, to any r.e. index for $\mathcal{S}$, any (u, l) in P and any Δ^0_α index for an instruction p for $\mathcal{S}$, a Δ^0_α index for a run s of the input (u, l, p) for $\mathcal{S}$ and an r.e. index for the corresponding $E(s)$. □

In Section 3 we relate this to the results of [2]. However, two comments on the choice of definitions seem appropriate here.

Comment 1. It may seem more natural to require in the definition that an instruction p should also assign to the empty sequence $\diamondsuit$ an element $u \in U$ for which $(u) \in P$, and to define a run *of* p in $\mathcal{S}$ to be a sequence $u_0, l_0, u_1, l_1, \ldots$ of which every initial segment is in P and such that, for each $n \geqslant 0$, $u_n = p(u_0, l_0, \ldots, u_{n-1}, l_{n-1})$.

Let us asume these definitions for the remainder of this comment.

Then every Δ^0_α instruction for an α-system $\mathcal{S}$ has a conservative Δ^0_α run, provided that $\mathcal{S}$ satisfies the additional condition:

$(4)^+$ For each $(u) \in P$ there exists l for which $(u, l) \in P$.

This statement follows immediately from Proposition 1. In this case, however, it will not necessarily be possible to compute an r.e. index for $E(s)$ only from a Δ^0_α index for p. For this additional uniformity, which is in fact convenient in Section 6, one may deduce the necessary further assumption by adding a new u^* and l^* to the beginning of each sequence in P and suitably extending the system. Condition (4) for the resulting system then gives the following additional condition on the present system.

$(4)^{++}$ If $(u) \in P$, if $\alpha > \xi_1 > \xi_2 > \cdots > \xi_k \geqslant 0$ and if $m_2 \subseteq_{\xi_2} m_3 \subseteq_{\xi_3} \cdots \subseteq_{\xi_{k-1}} m_k$, then there exists l for which $(u, l) \in P$ and $m_i \subseteq_{\xi_i} l$ for each $i = 2, 3, \ldots, k$.

To avoid this complication it seems preferable to specify, in addition to the instruction p, the first entry, u, of the desired run, rather than just a Δ^0_α index for it. By further specifying its second entry l, we also avoid adding condition (4^+).

Comment 2. For the sake of applications, this least restrictive method of defining the sets P and the instructions for α-systems is presumably the most useful. On the other hand, we note that for any α-system $\mathcal{S} = (U, L, P, E, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ we may define an 'equivalent' α-system $\mathcal{S}' = (U', L', P', E', \langle \subseteq'_\xi \rangle_{\xi<\alpha})$ for which the condition $(u'_0, l'_0, u'_1, l'_1, \ldots) \in P$ depends only on the sets of pairs (u'_i, l'_i) and (l'_i, u'_{i+1}) and such that instructions p for $\mathcal{S}$ correspond to instructions P' for $\mathcal{S}'$ with the property that $p'(u'_0, l'_0, \ldots, u'_n, l'_n)$ depends only on l'_n. Thus, if desired, these properties of P and p may be assumed without loss of generality.

The definition of $\mathcal{S}'$ is as follows. Let U' and L' consist of those sequences $(u_0, l_0, \ldots, u_n)$ and $(u_0, l_0, \ldots, u_n, l_n)$ from P respectively where $n \geqslant 0$. Let P' consist of those finite sequences $(u'_0, l'_0, u'_1, l'_1, \ldots)$ for which each $u'_i \in U'$, each $l'_i \in L'$, each sequence l'_i is a 1-place extension of the sequence u'_i and each u'_{i+1} is a 1-place extension of l'_i. If $l' = (\ldots, l) \in L'$ define $E'(l') = E(l)$ and if also $m' = (\ldots, m) \in L'$ let $l' \subseteq'_\xi m'$ iff $l \subseteq_\xi m$.

Then for any input (u, l, p) for $\mathcal{S}$ we obtain a corresponding input (u', l', p')

for $\mathcal{S}'$ where $u' = (u)$, $l' = (u, l)$ and for each $(u_0', l_0', \ldots, u_n', l_n') \in P'$, $p'(u_0', l_0', \ldots, u_n', l_n') = p(l_n')$.

We emphasize that Proposition 1 is true for all $\alpha \geq 1$ *including the cases where* α *is a limit ordinal.* In these cases the result does have a different appearance from the treatment in [2] where only a special sort of limit system was considered. It seems likely that this latter sort of system will no longer be useful, but in case it is, and also to assist with the discussions of Sections 3 and 4, we give here the corresponding definitions and the corresponding result in the present format.

Definitions. Let α be a limit ordinal and let $\langle \alpha_n \rangle$ be a recursive increasing sequence of successor ordinals whose supremum is α.

An $\langle \alpha_n \rangle$*-system* is a structure $\mathcal{S} = (U, L, P, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ satisfying conditions (1), (2) and (3) for an α-system together with:

$(4)_0$ If $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}) \in P$, if $\alpha_n > \xi_1 > \xi_2 > \cdots > \xi_k$, and if $l_n = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \cdots \subseteq_{\xi_{k-1}} m_k$, then there exists l_{n+1} for which $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}, l_{n+1}) \in P$ and $m_i \subseteq_{\xi_i} l_{n+1}$ for each $i = 1, 2, \ldots, k$.

[Thus, in comparison with the previous condition (4), α has been changed to α_n.]

An $\langle \alpha_n \rangle$*-input* for $\mathcal{S}$ is an input (u, l, p) for which each restriction of p to arguments of the form $(u_0, l_0, \ldots, u_n, l_n)$ *is* $\Delta^0_{\alpha_n}$ uniformly in n, and an $\langle \alpha_n \rangle$*-index* for such a p is a recursive index for a corresponding sequence of $\Delta^0_{\alpha_n}$ indices.

Proposition 2. *Every $\langle \alpha_n \rangle$-input for an $\langle \alpha_n \rangle$-system has a conservative Δ^0_α run.*

Moreover, there is a partial effective procedure which assigns to u, l, any $\langle \alpha_n \rangle$-index for p and any r.e. index for $\mathcal{S}$, a Δ^0_α index for a conservative run s of the input (u, l, p) for $\mathcal{S}$ and an r.e. index for $E(s)$. $\square$

Comment. Thus, from assumption $(4)_0$ which is weaker than (4), we obtain the weaker conclusion that $\langle \alpha_n \rangle$-inputs, rather than arbitrary Δ^0_α inputs, are accepted. We could add that the nature of the Δ^0_α run $u_0, l_0, u_1, l_1, \ldots$ obtained is special in the same sense that recursively in n one may obtain a $\Delta^0_{\alpha_n}$ index for u_{n+1} and l_{n+1}.

Both Propositions 1 and 2 follow, as described in Section 3, from the construction of [2].

3. New systems from old

In contrast with the notion of an α-system described in Section 2, we shall refer to the α-systems of [2] of the form $\mathcal{T} = (T, L, S, N, F, \langle \lhd_\xi \rangle_{1 \leq \xi < \alpha})$ as *old* α-systems.

These were defined only when α is a successor ordinal and for limit ordinals only old $\langle \alpha_n \rangle$-systems were considered, analogous to our present $\langle \alpha_n \rangle$-systems. The definition of an old α-system involved seven conditions and we should immediately note that the first of these is given incorrectly in [2] and should instead read:

(1) For each $u \in T_0$ there exists $l \in L$ such that $S(u, l)$ and $F(l) \neq \emptyset$.

We assume, in this section only, familiarity with [2] and relate the old systems to the new systems in two steps. In Step 1, we describe how a quite minor modification of the definition of old α-systems does not affect the argument given in [2]. In Step 2 we show how a new α-system can be converted into a modified old α-system, thereby establishing Proposition 1 when α is a successor ordinal.

In just the same way, we can establish Proposition 2 for our new $\langle \alpha_n \rangle$-systems.

We conclude the section by showing that the results for the old α-systems may quickly be deduced from the results for new ones.

Step 1. Let α be a successor ordinal. By a *modified* old α-system, we mean a system $\mathcal{T} = (T, L, S, N, F, \langle \lhd_\xi \rangle_{\xi<\alpha}, E)$ for which $(T, L, S, N, F, \langle \lhd_\xi \rangle_{1\leq\xi<\alpha})$ is an old α-system over some complete recursive metric space X, $E \subseteq L \times \mathbb{N}$ is r.e., $\lhd_0$ is an extra r.e. binary relation on L and the following further conditions are fulfilled.

(8) $\lhd_0$ is reflexive and transitive.
(9) If $l \lhd_0 m$, then $E(l) \subseteq E(m)$ [where $E(l) = \{m : (l, m) \in E\}$].
(5)* As for (5) of [2] with $1 \leq \gamma_1$ replaced by $0 \leq \gamma_1$.
(7)* As for (7) of [2] with $\alpha_0 \geq 1$ replaced by $\alpha_0 \geq 0$.

Proposition 1 of [2] is established using a 'Main Lemma' in which a '1-precursor' $\mathcal{T}' = (T', L', S', N', F')$ of $\mathcal{T}$ is constructed. Because of the stronger version (7)* of condition (7), we can, in exact analogy with the construction of β-precursors for $\beta > 1$, define the relation $\lhd_0'$ on L', based on $\lhd_0$, and require that whenever $N'(u', l', v', m')$ for $u', v' \in T'$ and $l', m' \in L'$, then $l' \lhd_0' m'$.

As in [2], for each $u \in T_0$, each $l \in L$ with $S(u, l)$ and $F(l) \neq \emptyset$, and each Δ^0_α instruction p, there exist coresponding u', l' and p' for $\mathcal{T}'$, where p' is recursive. There is thus a recursive labelling u_0', l_0', u_1', l_1', ... of p' in $\mathcal{T}'$, beginning with u', l' and having an r.e. adherent point, and this in turn, by a repeated limiting process, gives a Δ^0_α labelling u_0, l_0, u_1, l_1, ... of p beginning with u, l and having the same adherent point.

But we may make a further deduction. Each l_i is, by the construction, a hereditarily finite sequence from L and, again by the construction, the result, $\hat{l}_i$, of removing the sequence-forming symbols and deleting repetitions is a finite sequence from L in which every two adjacent members are related, in order, by $\lhd_\xi$ for some $\xi \geq 1$. In just the same way, the result of removing repetitions from the concatenated sequence $\hat{l}_0, \hat{l}_1, \hat{l}_2, \ldots$ gives subsequences $\hat{\hat{l}}_i$ for which the last entry of $\hat{\hat{l}}_i$ is in the relation $\lhd_0$ to the first entry of $\hat{\hat{l}}_{i+1}$.

Thus, by condition (5), in the concatenated sequence $\sigma = \hat{l}_0, \hat{\hat{l}}_1, \hat{\hat{l}}_2, \ldots$, adjacent pairs are related, in order, by $\lhd_0$. Now the Δ^0_α sequence $l_0, l_1, \ldots$ is obtained by a repeated limiting process from $l'_0, l'_1, \ldots$ and is therefore a subsequence of σ. On the other hand, the sequence m_i of last elements of the $\hat{\hat{l}}_i$, which is certainly recursive, is also a subsequence of σ. By condition (9) it follows that $\bigcup_i E(l_i) = \bigcup_i E(m_i)$ which is r.e.

We have thus shown that for a modified old α-system $\mathcal{T}$, for each $u \in T_0$, each l with $S(u, l)$ and $F(l) \neq \emptyset$, and eeach Δ^0_α instruction p, there is a Δ^0_α labelling $u_0, l_0, u_1, l_1, \ldots$ of p where $u_0 = u$, $l_0 = l$, having an r.e. adherent point and for which also $\bigcup_n E(l_n)$ is r.e.

Step 2. We continue to assume that α is a successor ordinal, say $\alpha = \beta + 1$, and we now observe that any (new) α-system $\mathcal{T} = (U, L, P, E, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ can be converted to a modified old α-system $\mathcal{T}' = (T', L', S', N', F', \langle \lhd_\xi \rangle_{\xi<\alpha}, E')$ over any complete recursive metric space X.

For $n \geqslant 0$, let T'_n be the set of all sequences $(u_0, l_0, \ldots, u_{n-1}, u_n) \in P$ for which each $l_i \lhd_\beta l_{i+1}$. For $n > 0$, the predecessor of $(u_0, l_0, \ldots, u_{n-1}, l_n, u_n)$ in T' is, of course, $(u_0, l_0, \ldots, u_{n-1})$. Let L' be the set of all sequences $(u_0, l_0, \ldots, u_n, l_n) \in P$ for $n \geqslant 0$ for which, also, each $l_i \lhd_\beta l_{i+1}$.

Define S' to hold precisely for those $(u', l') \in T' \times L'$ for which the sequence l' extends the sequence u' by exactly one entry and similarly let N' hold for those $(u', l', v', m') \in T' \times L' \times T' \times L'$ for which l', v', m' each extend the sequences u', l', v' respectively by one entry.

For each $l' \in L'$, define $F'(l') = X$. If $l' = (u_0, l_0, \ldots, u_n, l_n)$, define $E'(l') = E(l_n)$ and if also $m' = (v_0, m_0, \ldots, v_k, m_k)$ define $l' \lhd_\xi m'$ iff $l_n \subseteq_\xi m_k$.

Now, for any Δ^0_α input (u, l, p) for $\mathcal{T}$, p will determine an instruction p' for $\mathcal{T}'$ by $p'(u', l') = (u_0, l_0, \ldots, u_n, l_n, u_{n+1})$ where, if $l' = (u_0, l_0, \ldots, u_n, l_n)$ then $u_{n+1} = p(u_0, l_0, \ldots, u_n, l_n)$. By the conclusion of Step 1, there is a Δ^0_α labelling $u'_0, l'_0, u'_1, l'_1, \ldots$ of p' with $u'_0 = (u)$, $l'_0 = (u, l)$ and for which $\bigcup_n E'(l_n)$ is r.e. But by the definition of $\mathcal{T}'$, these are the initial segments of a Δ^0_α conservative run of the input (u, l, p) on $\mathcal{T}$. This establishes Proposition 1 in the case where α is a successor ordinal.

$\langle \alpha_n \rangle$-systems

Steps 1 and 2 may be used in the same way to establish Proposition 2 of Section 2. The only alteration to Step 1 is that for a modified old $\langle \alpha_n \rangle$-system condition (7)* results from the same change to condition (7)′ of [2] rather than to condition (7). For Step 2, the definition of U' is the set of sequences $(u_0, l_0, \ldots, u_i) \in P$ such that each $l_n \lhd_{\beta_n} l_{n+1}$ where $\alpha_n = \beta_n + 1$, and the definition of L' is similarly altered.

Re-designing old systems

When, in Step 2, a new α-system is treated as an old one, we take each $F(l)=X$, so the metric space X has no further significance. In spite of this, there is actually no loss of generality in using new α-systems rather than old ones, since the construction of an r.e. adherent point of X may be incorporated into a suitable choice of the relation E, as follows.

Let $\mathscr{T}=(T, L, S, N, F, \langle \lhd_\xi \rangle_{\xi<\alpha})$ be an old α-system over a complete metric space X with a family $B(X)$ of chosen non-empty basic open sets. We define an 'equivalent' new α-system $\mathscr{T}'=(U', L', P', E', \langle \subseteq'_\xi \rangle_{\xi<\alpha})$. Let $U'=U$ and let L' be the set of all pairs $(l, \sigma) \in L \times B(X)$ for which $F(l) \cap \sigma \neq \emptyset$. Let P' consist of those finite sequences $(u_0, (l_0, \sigma_0), u_1, (l_1, \sigma_1), \ldots)$ for which each $u_i \in U'$, each $(l_i, \sigma_i) \in L'$, $S(u_0, l_0)$, each $N(u_i, l_i, u_{i+1}, l_{i+1})$, each $\sigma_{i+1} \prec \sigma_i$ and each $\delta(\sigma_i) < 1/(i+1)$. If $l'=(l, \sigma) \in L'$ define $E'(l')=\{r \in B(X): \sigma \prec \tau\}$ and if also $m'=(m, \tau) \in L'$ define $l' \subseteq'_\xi m'$ if $\sigma \prec \tau$ and, when $\xi \geq 1$, $l \lhd_\xi m$.

Now for any $u \in T_0$ and any Δ^0_α instruction p for old α-system $\mathscr{T}$, let $l \in L$ be such that $(u, l) \in S$ and $F(l) \neq \emptyset$ and let $\sigma \in B(X)$ be such that $F(l) \cap \sigma \neq \emptyset$. Define the instruction p' for $\mathscr{T}'$ by $p'(u_0, (l_0, \sigma_0), u_1, (l_1, \sigma_1), \ldots, u_n, (l_n, \sigma_n)) = p(u_n, l_n)$. Then $(u, (l, \sigma), p')$ is an input for $\mathscr{T}'$. If $(u_0, (l_0, \sigma_0), u_1, (l_1, \sigma_1), \ldots)$ is a conservative Δ^0_α run of this instruction on $\mathscr{T}'$, then $(u_0, l_0, u_1, l_1, \ldots)$ is a Δ^0_α labelling of p in $\mathscr{T}$ having the adherent point $x \in X$ where $\{x\} = \bigcap_n \sigma_n$. Clearly $\{\tau \in B(X): x \in \tau\} = \bigcup_n E(l_n, \sigma_n)$ and so x is an r.e. point of X. Thus, instead of the old α-system $\mathscr{T}$, we may consider the new α-system $\mathscr{T}'$.

Comment 1. The same construction shows how a *modified* old α-system $\mathscr{T}=(T, L, S, N, F, \langle \lhd_\xi \rangle_{\xi<\alpha}, E)$ can also be replaced by a new α-system $\mathscr{T}'=(U', L', P', E', \langle \subseteq'_\xi \rangle_{\xi<\alpha})$. The only differeces are that now $(l, \sigma) \subseteq'_0 (m, \tau)$ iff $\sigma \prec \tau$ and $l \lhd_\xi m$ and $E'(l, \sigma) = E(l) \times \{\tau \in B(X): \sigma \prec \tau\}$.

Comment 2. The same construction again also shows how an old $\langle \alpha_n \rangle$-system, modified or not, can be replaced by a new $\langle \alpha_n \rangle$-system.

Comment 3. In view of Step 2 and Comment 1, it is clear that our new α-systems are precisely as general as modified old α-systems. But, since the former seem to be more natural, more simply described and more simply applied, there seems at present little reason to pursue the metric space approach.

4. Limit α-systems

In Section 3 we established Proposition 1 for successor ordinals and Proposition 2, by modifying the corresponding arguments from [2]. We now show how Proposition 1 for limit ordinals can be deduced directly from Proposition 2, without reference to [2].

Let $\mathcal{T} = (U, L, P, E, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ be a (new) α-system where α is a limit ordinal. Let $\langle \alpha_n \rangle$ be an increasing recursive sequence of successor ordinals, whose supremum is α. We may construct an 'equivalent' (new) $\langle \alpha_n \rangle$-system $\mathcal{T}' = (U', L', P', E', \langle \subseteq'_\xi \rangle_{\xi<\alpha})$ as follows.

Let $U^* = \{u^* : u \in U\}$ be a disjoint copy of U. We use the notation $u^{(*)}$ for $u \in U$ to mean either of u or u^*.

Let U' consist of those sequences $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^{(*)})$ for which $i \geqslant 0$ and $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}) \in P$. Let L' consist of those sequences $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^{(*)}, l_{i+1})$ for whch $i \geqslant 0$ and $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}) \in P$.

Let P' consist of those sequences $(u'_0, l'_0, u'_1, l'_1, \ldots)$ where each $u'_i \in U'$, $l'_i \in L'$ for which:

(i) u'_0 is of the form (u_0, l_0, u_1^*).

(ii) l'_0 is of the form (u_0, l_0, u_1^*, l_1) where $l_0 \subseteq_{\alpha_0} l_1$.

(iii) If l'_n is $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*, l_{i+1})$, then u'_{n+1} is either $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*)$ or $(u_0, l_0, \ldots, u_i, l_i, u)$ for some $u \in U$.

(iv) If l'_n is $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1})$, then u'_{n+1} is $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}, u^{(*)})$ for some $u \in U$.

(v) If u'_n is $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^{(*)})$, then l'_n is $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}^{(*)}, l)$, for some $l \in L$ such that $l_i \subseteq_{\alpha_n} l$.

For $l' = (\ldots, l) \in L'$ we define $E'(l') = E(l)$. If also $m' = (\ldots, m) \in L'$, then we define $l' \subseteq'_\xi m'$ iff $l \subseteq_\xi m$.

Comment. The purpose of the u^*'s is that they are arbitrary choices of u's which are repeatedly re-labelled in the sequence of l'_n's until n becomes sufficiently large that the correct choice of u has become defined.

To show that $\mathcal{T}'$ is an $\langle \alpha_n \rangle$-system, we must verify condition $(4)_0$ of Section 1, as follows.

Suppose that $(u'_0, l'_0, \ldots, u'_n, l'_n, u'_{n+1}) \in P'$, that $\alpha_n > \xi_1 > \xi_2 > \cdots > \xi_k$ and that $l'_n = m'_1 \subseteq'_{\xi_1} m'_2 \subseteq'_{\xi_2} \cdots \subseteq'_{\xi_{k-1}} m'_k$. Then we must find l'_{n+1} for which $(u'_0, l'_0, \ldots, u'_n, l'_n, u'_{n+1}, l'_{n+1}) \in P$ and $m'_j \subseteq'_{\xi_j} l'_{n+1}$ for $j = 1, 2, \ldots, k$.

We may let each $m'_j = (\ldots, m_j)$.

Case 1: $l'_n = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*, l_{i+1})$ where $l_i \subseteq_{\alpha_n} l_{i+1}$.

Case 1(a): $u'_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*)$. In this case we need $l'_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*, l)$ where $l_i \subseteq_{\alpha_{n+1}} l$ and each $m_j \subseteq_{\xi_j} l$. But this is possible by condition (4) for $\mathcal{T}$ since we have $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}) \in P$, $\alpha > \alpha_{n+1} > \alpha_n > \xi_1 > \cdots > \xi_k$ and $l_i \subseteq_{\alpha_{n+1}} l_i \subseteq_{\alpha_n} l_{i+1} = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \cdots \subseteq_{\xi_{k-1}} m_k$.

Case 1(b): $u_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u)$. In this case we need $l'_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u, l)$ where $l_i \subseteq_{\alpha_{n+1}} l$ and each $m_j \subseteq_{\xi_j} l$. The argument is the same as for 1(a).

Case 2: $l'_n = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1})$ where $l_i \subseteq_{\alpha_n} l_{i+1}$ and $u'_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}, u^{(*)})$.

In this case we need $l'_{n+1} = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}, u^{(*)}, l)$ where

$l_{i+1} \subseteq_{\alpha_{n+1}} l$ and each $m_j \subseteq_{\xi_j} l$. This is again possible by condition (4) for $\mathcal{T}$ since $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}, u_{i+2}) \in P$ and $l_{i+1} \subseteq_{\alpha_{n+1}} l_{i+1} = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \cdots \subseteq_{\xi_{k-1}} m_k$.

Thus $\mathcal{T}'$ is indeed an $\langle \alpha_n \rangle$-system. It remains to see how a Δ^0_α input (u, l, p) for $\mathcal{T}$ can be converted into a suitable $\langle \alpha_n \rangle$-input (u', l', p') for $\mathcal{T}'$.

We may choose an arbitrary (u, l, u_1, l_1) in P for which $l \subseteq_{\alpha_0} l_1$ and define $u' = (u, l, u_1^*)$ and $l' = (u, l, u_1^*, l_1)$.

Now since p is a Δ^0_α function, we have $p = \bigcup_n p_n$ where uniformly in n each p_n is a partial $\Delta^0_{\alpha_n}$ function. We may define $\hat{p}_n(x) = y$ if any one of $p_0(x), \ldots, p_n(x)$ converges to y in at most n steps, so the domain of $\hat{p}_n$ is $\Delta^0_{\alpha_n}$, uniformly in n, and $p = \bigcup_n \hat{p}_n$.

We define the $\langle \alpha_n \rangle$-instruction p' for $\mathcal{T}'$ as follows. Let $\sigma = (u_0', l_0', \ldots, u_n', l_n') \in P$. If $l_n' = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*, l_{i+1})$, then $p'(\sigma) = (u_0, l_0, \ldots, u_i, l_i, u)$ if $\hat{p}_n(u_0, l_0, \ldots, u_i, l_i) = u$ and $p'(\sigma) = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}^*)$ if $\hat{p}_n(u_0, l_0, \ldots, u_i, l_i)$ is not defined. If $l_n' = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1})$, then $p'(\sigma) = (u_0, l_0, \ldots, u_i, l_i, u_{i+1}, l_{i+1}, u)$ if $\hat{p}_n(u_0, l_0, \ldots, u_{i+1}, l_{i+1}) = u$ and, if $\hat{p}_n(u_0, l_0, \ldots, u_{i+1}, l_{i+1})$ is not defined, then $p'(\sigma) = (u_0, l_0, \ldots, u_i, l_{i+1}, u_{i+2}^*)$ for the first available sequence of the form $(u_0, l_0, \ldots, u_{i+1}, l_{i+1}, u_{i+2})$ in P.

By Proposition 2, there is a conservative $\langle \alpha_n \rangle$ run, s', of the input (u', l', p') for $\mathcal{T}'$. Let $s' = (u_0', l_0', u_1', l_1', \ldots)$. If we let l_n'' denote the result of removing from the sequence l_n' the entry of the form u^*, if any, and the following entry from L, then the sequences $l_0'', l_1'', l_2'', \ldots$ will form non-decreasing initial segments of a run, s, on $\mathcal{T}$, of the input p for $\mathcal{T}$, and since all members of L appearing in the l_n' are linearly-ordered by $\subseteq_0$, we deduce that $E(s) = E'(s')$ which is r.e. Thus there is a conservative Δ^0_α run of (u, l, p) on $\mathcal{T}$, which establishes Proposition 1 in the remaining case where α is a limit ordinal.

5. Infinitary formulas

In the case of r.e. structures, we wish to define the Σ_α and Π_α formulas of $L_{\omega_1\omega}$ in such a way that the *recursive* Σ_α and Π_α formulas form suitable classes of formulas which on any r.e. structure will define, respectively, Σ^0_α and Π^0_α relations. To avoid confusion with the corresponding notions in [2] we therefore use the superscript $+$, since in r.e. structures only the unnegated atomic formulas together with the formulas $x_i \neq x_j$ will necessarily be r.e.

Definitions. A *basic positive formula* is a finite conjunction of formulas each of which is either an unnegated atomic formula or a formula of the form $x_i \neq x_j$.

A basic *negative* formula is similarly a finite disjunction of formulas each of which is either a negated atomic formula or a formula of the form $x_i = x_j$.

[Thus, both the basic positive and basic negative formulas contain formulas which are identically false and formulas which are identically true.]

The Σ_0^+ and Π_0^+ formulas are precisely the truth values T and F.

For $\alpha > 0$, the Σ_α^+ formulas are those of the form

$$\bigvee\!\!\!\bigvee_n \exists \bar{y}_n \, (\phi_n(\bar{x}, \bar{y}_n) \mathbin{\&} \psi_n(\bar{x}, \bar{y}))$$

where $\bar{x}$ and each $\bar{y}_n$ denote finite sequences of variables, each ϕ_n is a basic positive formula and each ψ_n is Π_β^+ for some $\beta < \alpha$. Likewise, the Π_α^+ formulas are those of the form

$$\bigwedge\!\!\!\bigwedge_n \forall \bar{y}_n \, (\phi_n(\bar{x}, \bar{y}_n) \vee \psi_n(\bar{x}, \bar{y}_n))$$

where each ϕ_n is basic negative and each ψ_n is Σ_β^+ for some $\beta < \alpha$.

Notes. While the Σ_1^+ formulas are clearly the positive Σ_1 formulas, subject to the convention that occurrences of $\neq$ are regarded as positive, the Σ_2^+ formulas, for example, need not, of course, be positive since they include all negated atomic formulas.

We also note that all basic positive formulas are Π_2^+, so that for $\alpha > 2$ the definition of Σ_α^+ formulas could be given more simply to be those formulas of the form $\bigvee\!\!\!\bigvee_n \exists \bar{y}_n \, \psi_n(\bar{x}, \bar{y}_n)$ for which each ψ_n is Π_β^+ for some $\beta < \alpha$, and the definition of Π_α^+ formulas for $\alpha > 2$ could analogously be simplified. We have used the present definition in order to avoid repetition in the proofs of the lemmas.

The Σ_α and Π_α formulas (without the sign $+$) have a similar, more straightforward definition, given in [2]. For the purposes of this paper, since these classes are occasionally referred to, we may use the following definition which has the same effect up to logical equivalence.

Definition. The Σ_α and Π_α formulas are those obtained from Σ_α^+ and Π_α^+ formulas by arbitrary substitutions of negated atomic formulas for occurrences of atomic formulas or vice versa.

Notation. For any structure $\mathfrak{A}$, we let $\Sigma_\alpha^+(\mathfrak{A})$ denote the set of all Σ_α^+ sentences true in $\mathfrak{A}$, and similarly for $\Pi_\alpha^+(\mathfrak{A})$.

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, not necessarily different, and let $\bar{a}$ and $\bar{b}$ be finite sequences from $\mathfrak{A}$ and $\mathfrak{B}$ respectively, having the same lengths. We proceed to obtain an equivalent formulation of the statement that $\Pi_\alpha^+(\mathfrak{A}, \bar{a}) \subseteq \Pi_\alpha^+(\mathfrak{B}, \bar{b})$.

Definition. For sequences $\bar{a}$ from $\mathfrak{A}$ and $\bar{b}$ from $\mathfrak{B}$ of the same lengths, we define $\bar{a} \leq_0^+ \bar{b}$ to be invariably true. For $\alpha > 0$ we define $\bar{a} \leq_\alpha^+ \bar{b}$ if for every sequence $\bar{d}$

from $\mathfrak{B}$, every $\beta < \alpha$ and every basic positive formula ϕ true for $\bar{b}, \bar{d}$ in $\mathfrak{B}$, there is a sequence $\bar{c}$ from $\mathfrak{A}$, of the same lengths as $\bar{d}$, for which $\bar{a}, \bar{c} \geqslant^+_\beta \bar{d}, \bar{d}$ and ϕ is true for $\bar{a}, \bar{c}$ in $\mathfrak{A}$.

Note. In the event that elements may belong to more than one of the structures under consideration, this notation, while compact, is imprecise and should perhaps be expanded to $(\mathfrak{A}, \bar{a}) \leqslant^+_\alpha (\mathfrak{B}, \bar{b})$. But then, in view of the next lemma, we may as well write $\Pi^+_\alpha(\mathfrak{A}, \bar{a}) \subseteq \Pi^+_\alpha(\mathfrak{B}, \bar{b})$.

Lemma 1. *For sequences $\bar{a}$ from $\mathfrak{A}$ and $\bar{b}$ from $\mathfrak{B}$ and for all α, the following are equivalent.*

(i) $\bar{a} \leqslant^+_\alpha \bar{b}$,

(ii) $\Pi^+_\alpha(\mathfrak{A}, \bar{a}) \subseteq \Pi^+_\alpha(\mathfrak{B}, \bar{b})$,

(iii) $\Sigma^+_\alpha(\mathfrak{A}, \bar{a}) \supseteq \Sigma^+_\alpha(\mathfrak{B}, \bar{b})$.

Proof. Clearly (ii) and (iii) are equivalent, since the negation of any Σ^+_α formula is logically equivalent to a Π^+_α formula, and vice versa.

For $\alpha = 0$, all the conditions are trivially true from the definitions. We show the equivalence of (i) and (iii) by transfinite induction on $\alpha > 0$.

Suppose first that $\bar{a} \leqslant^+_\alpha \bar{b}$. Let $\psi(\bar{x})$ be any Σ^+_α formula true for $\bar{b}$ in $\mathfrak{B}$. Then $\psi(\bar{x})$ is of the form

$$\bigvee\!\!\!\bigvee_n \exists \bar{y}_n\, (\phi_n(\bar{x}, \bar{y}_n) \,\&\, \psi_n(\bar{x}, \bar{y}_n))$$

where ϕ_n is basic positive and ψ_n is $\Pi^+_{\beta_n}$ for some $\beta_n < \alpha$. Thus, for some n and some $\bar{d}$ from $\mathfrak{B}$, the formulas ϕ_n and ψ_n are true for $\bar{b}, \bar{d}$ in $\mathfrak{B}$. By the definition of $\leqslant^+_\alpha$, there therefore exists $\bar{c}$ from $\mathfrak{A}$ for which $\bar{a}, \bar{c} \geqslant^+_\beta \bar{b}, \bar{d}$ and ϕ_n is true in $\mathfrak{A}$ for $\bar{a}, \bar{c}$. But then, by induction hypothesis, $\Pi^+_{\beta_n}(\mathfrak{A}, \bar{a}, \bar{c}) \supseteq \Pi^+_{\beta_n}(\mathfrak{B}, \bar{b}, \bar{d})$ and so also ψ_n is true in $\mathfrak{A}$ for $\bar{a}, \bar{c}$. Thus ψ is true for $\bar{a}$ in $\mathfrak{A}$. We have therefore shown that $\Sigma^+_\alpha(\mathfrak{A}, \bar{a}) \supseteq \Sigma^+_\alpha(\mathfrak{B}, \bar{b})$ as required for (iii).

Conversely, suppose that $\Sigma^+_\alpha(\mathfrak{A}, \bar{a}) \supseteq \Sigma^+_\alpha(\mathfrak{B}, \bar{b})$. We wish to show that $\bar{a} \leqslant^+_\alpha \bar{b}$, so let $\bar{d}$ be a sequence from $\mathfrak{B}$, let $\beta < \alpha$ and let $\phi(\bar{x}, \bar{y})$ be a basic positive formula true in $\mathfrak{B}$ for $\bar{c}, \bar{d}$. Consider all those sequences $\bar{a}_i, \bar{c}_i$ from $\mathfrak{A}$ having the same lengths as $\bar{b}, \bar{d}$ and for which $\bar{a}_i, \bar{c}_i \not\geqslant^+_\beta \bar{b}, \bar{d}$. For each i, by induction hypothesis, we may choose a Π^+_β formula $\psi_{\beta,i}(\bar{x}, \bar{y})$ which is true in $\mathfrak{B}$ for $\bar{b}, \bar{d}$ but false in $\mathfrak{A}$ for $\bar{a}_i, \bar{c}_i$. Let ψ_β be a Π^+_β formula equivalent to $\bigwedge\!\!\!\bigwedge_i \psi_{\beta,i}$. Then the formula $\exists \bar{y}\, (\phi(\bar{x}, \bar{y}) \,\&\, \psi_\beta(\bar{x}, \bar{y}))$ is a Σ^+_α formula true of $\bar{b}$ in $\mathfrak{B}$ and so, by supposition, true of $\bar{a}$ in $\mathfrak{A}$. So there exists $\bar{c}$ from $\mathfrak{A}$ such that both ϕ and ψ_β are true for $\bar{a}, \bar{c}$ in $\mathfrak{A}$. It follows that $\bar{a}, \bar{c} \geqslant^+_\beta \bar{b}, \bar{d}$, since otherwise $\bar{a}, \bar{c}$ would be $\bar{a}_i, \bar{c}_i$ for some i, and ψ_β is false for each $\bar{a}_i, \bar{c}_i$. We have therefore shown that $\bar{a} \leqslant^+_\alpha \bar{b}$. $\square$

Note. As previously remarked, the definitions of the classes of Σ^+_α and Π^+_α formulas for $\alpha > 2$ need not refer to basic positive formulas. In view of Lemma 1, neither needs the definition of $\bar{a} \leqslant_{\alpha}^{+} \bar{b}$ for $\alpha > 2$, which can equivalently be given as: for all $\bar{d}$ and all $\beta < \alpha$ there exists $\bar{c}$ for which $\bar{a}, \bar{c} \geqslant_{\beta}^{+} \bar{b}, \bar{d}$.

In our constructions, we make use of a fixed infinite recursive set, C, a family of structures, $\mathfrak{A}_i$, and various finite partial one–one functions from C into one or other of the $\mathscr{A}_i$.

Definition. If f, g are finite partial one–one functions from C into $\mathfrak{A}_i$, $\mathfrak{A}_j$ respectively, then we define $f \lhd_{\alpha}^{+} g$ if $\operatorname{dom}(g) \supseteq \operatorname{dom}(f) = \bar{d}$, say, and $f(\bar{d}) \leqslant_{\alpha}^{+} g(\bar{d})$.

Note. As for the definition of the relations $\leqslant_{\alpha}^{+}$, if it is ambiguous which structures are to be regarded as the codomains of f and g, then one should perhaps write $(\mathfrak{A}_i, f) \lhd_{\alpha}^{+} (\mathfrak{A}_j, g)$.

The significance of this definition, for our purposes, lies in the following lemmas.

Lemma 2. *Suppose that f, g are as above and that $f \lhd_{\alpha}^{+} g$. Suppose further that $\alpha > \beta$, that $\bar{c}$ is a sequence from C and that $\phi(\bar{x})$ is a basic positive sentence true for $g(\bar{c})$ in $\mathfrak{A}_j$. Then there exists a finite partial one–one $h \supseteq f$ for which ϕ is true for $h(\bar{c})$ in $\mathfrak{A}_i$ and $g \lhd_{\beta}^{+} h$.*

Proof. We may suppose that $\operatorname{dom}(g) = \bar{c} = \bar{c}_1, \bar{c}_2$ where $\bar{c}_1 = \operatorname{dom}(f)$ and that $f(\bar{c}_1) = \bar{a}_1$, $g(\bar{c}_1) = \bar{b}_1$ and $g(\bar{c}_2) = \bar{b}_2$. Then since $f \lhd_{\alpha}^{+} g$, we have $\bar{a}_1 \leqslant_{\alpha}^{+} \bar{b}_1$. Let ϕ' be the conjunction of ϕ with all the inequalities which hold for $\bar{b}_1, \bar{b}_2$. Then ϕ' is also a basic positive formula and, since ϕ' is true for $\bar{b}_1, \bar{b}_2$ in $\mathfrak{A}_j$ then, by definition of $\leqslant_{\alpha}^{+}$, there exists $\bar{a}_2$ from $\mathfrak{A}_i$ for which $\bar{a}_1, \bar{a}_2 \geqslant_{\beta}^{+} \bar{b}_1, \bar{b}_2$ and ϕ' is true for $\bar{a}_1, \bar{a}_2$ in $\mathfrak{A}_i$. We may therefore define h by $h(\bar{c}_1) = \bar{a}_1$ and $h(\bar{c}_2) = \bar{a}_2$. $\square$

Lemma 3. *Suppose that, for each $i = 1, 2, \ldots, k$, f_i is a finite partial one–one function from C to $\mathfrak{A}_i$. Suppose further that $\xi_1 > \xi_2 > \cdots > \xi_k \geqslant 0$, that $f_1 \lhd_{\xi_1}^{+} f_2 \lhd_{\xi_2}^{+} \cdots \lhd_{\xi_{k-1}}^{+} f_k$, that $\bar{c}$ is a sequence from $\operatorname{dom}(f_k)$ and that $\phi(\bar{x})$ is a basic positive formula true for $f_k(\bar{c})$ in $\mathfrak{A}_k$. Then there is a finite partial one–one function g from C to $\mathfrak{A}_1$ for which $g \supseteq f_1$, ϕ is true for $g(\bar{c})$ in $\mathfrak{A}_1$ and $f_i \lhd_{\xi_i}^{+} g$ for $i = 1, 2, \ldots, k$.*

Proof. By induction on $k \geqslant 2$. For $k = 2$, the result follows from Lemma 2. For $k > 2$, by Lemma 2 there exists $h \supseteq f_{k-1}$ for which $f_k \lhd_{\xi_k}^{+} h$ and ϕ is true for $h(\bar{c})$ in $\mathfrak{A}_{k-1}$.

Then $f_{k-2} \lhd_{\xi_{k-2}}^{+} f_{k-1} \subseteq h$, so we have $f_1 \lhd_{\xi_1}^{+} f_2 \lhd_{\xi_2}^{+} \cdots \lhd_{\xi_{k-3}}^{+} f_{k-2} \lhd_{\xi_{k-2}}^{+} h$. Thus, by induction hypothesis, there exists $g \supseteq f_1$ such that $f_i \lhd_{\xi_i}^{+} g$ for $i = 1, 2, \ldots, k-2$, $h \lhd_{\xi_{k-1}}^{+} g$ and ϕ is true for $g(\bar{c})$ in $\mathfrak{A}_1$. But then $f_{k-1} \subseteq h \lhd_{\xi_{k-1}}^{+} g$, which gives $f_{k-1} \lhd_{\xi_{k-1}}^{+} g$ and $f_k \lhd_{\xi_k}^{+} h \lhd_{\xi_{k-1}}^{+} g$, which gives $f_k \lhd_{\xi_k}^{+} h$. $\square$

Results using Proposition 1, such as the theorem of the next section, appear to need that the relevant relations $\leqslant_\xi^+$ are uniformly r.e. Accordingly we make the following definitions, analogous to those of [5].

Definition. A *recursive sequence* of r.e. structures is a sequence $\langle \mathfrak{A}_i \rangle_{i \in I}$ of structures $\mathfrak{A}_i = (A_k \langle R_k^i \rangle_{k \in K})$ of the same recursive similarity type for which I is an r.e. set and each relation R_k^i is r.e. uniformly in i and k.

An *α-friendly* family of r.e. structures, where $\alpha < \omega_1^{\mathrm{CK}}$, is a recursive sequence $\langle \mathfrak{A}_i \rangle_{i \in I}$ of r.e. structures such that, using some notation for α, the relation $\Sigma_\xi^+(\mathfrak{A}_i, \bar{a}) \subseteq \Sigma_\xi^+(\mathfrak{A}_j, \bar{b})$ is r.e. in i, j, $\bar{a}$, $\bar{b}$ and $\xi < \alpha$.

Comment. In spite of the apparently restrictive nature of this condition, several commonly occurring classes of structures have this property. In fact, all of the examples given in [5] are clearly such, because for linearly ordered sets every Σ_ξ formula is equivalent to a Σ_ξ^+ formula and vice versa.

A separate but related phenomenon involves the notion of *recursive* Σ_ξ^+ and Π_ξ^+ formulas. These are the Σ_ξ^+ and Π_ξ^+ formulas respectively in which all the disjunctions and conjunctions appearing are r.e., in terms of a set of indices for recursive formulas defined simultaneously.

If, for structure $\mathfrak{A}$ and $\mathfrak{B}$, we have $\Pi_\xi^+(\mathfrak{A}) \neq \Pi_\xi^+(\mathfrak{B})$, then there is a Π_ξ^+ sentence ψ true in one structure but not the other. Even if $\mathfrak{A}$ and $\mathfrak{B}$ are r.e. structures, there seems no reason why there should be such a *recursive* Π_ξ^+ sentence ψ, but again for several commonly occurring classes of structures, this is the case. Lemmas 4 and 5 below establish one such set of circumstances.

Definitions. A *positive, open* formula is a finite disjunction of basic positive formulas. A *positive existential* formula is a (finitary) formula of the form $\exists \bar{y}\, \phi(\bar{x}, \bar{y})$ where ϕ is a positive open formula

The *positive open diagram* (respectively the *positive existential diagram*) *of a structure* $\mathfrak{A} = (A, \langle R_i \rangle_{i \in I})$ is the set of all positive open sentences (respectively positive existential sentences) of $L(A)$ which are true in $(\mathfrak{A}, \langle a \rangle_{a \in A})$.

Lemma 4. *Suppose that $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a recursive sequence of r.e. structures such that, using some notation for α, the relations $\Pi_\beta^+(\mathfrak{A}_i, \bar{a}) \subseteq \Pi_\beta^+(\mathfrak{A}_j, \bar{b})$ for $1 \leqslant \beta < \alpha$ are Σ_β^0 in i, j, $\bar{a}$, $\bar{b}$ and β. Suppose also that the positive existential diagram of each $\mathfrak{A}_i$ is recursive, uniformly in $i \in I$.*

Then for each $i \in I$, each sequence $\bar{a}$ from $\mathfrak{A}_i$ and each $\gamma \leqslant \alpha$ we can effectively find a recursive Π_γ^+ formula $\psi(\bar{x})$ such that, for all $j \in I$ and all sequences $\bar{b}$ from $\mathfrak{A}_j$ we have $\Pi_\gamma^+(\mathfrak{A}_i, \bar{a}) \subseteq \Pi_\gamma^+(\mathfrak{A}_j, \bar{b})$ iff $\mathfrak{A}_j \vDash \psi[\bar{b}]$.

Proof. For $\gamma = 0$, $\psi(\bar{x})$ may be taken to be the truth value T.

For $\gamma > 0$, we may proceed by recursive transfinite induction on $\gamma \leqslant \alpha$ to define, for each $\bar{a}$, i and γ, an index for a suitable formula ψ, in the following way. We note that, from the definition of the relations $\leqslant_\gamma^+$ and by Lemma 1, we have the following statement.

For all $\gamma > 0$, $\Pi_\gamma^+(\mathfrak{A}_i, \bar{a}) \subseteq \Pi_\gamma^+(\mathscr{A}_j, \bar{b})$ iff for every $\beta < \gamma$, every $\bar{d}$ from $\mathfrak{A}_j$ and every basic positive formula $\phi(\bar{x}, \bar{y})$ for which $\mathfrak{A}_j \models \phi[\bar{b}, \bar{d}]$, there exists $\bar{c}$ from $\mathfrak{A}_i$ for which $\Pi_\beta^+(\mathfrak{A}_i, \bar{a}, \bar{c}) \supseteq \Pi_\beta^+(\mathfrak{A}_j, \bar{b}, \bar{d})$ and $\mathfrak{A}_i \models \phi[\bar{a}, \bar{c}]$.

For fixed i, $\bar{a}$ and $\gamma > 0$ and for each $\beta < \gamma$, let S_β denote the set of all quadruples $\sigma = (j, \bar{b}, \bar{d}, \phi)$ for which $\bar{b}$ and $\bar{d}$ are sequences from $\mathfrak{A}_j$, $\bar{b}$ has the same length as $\bar{a}$, $\phi(\bar{x}, \bar{y})$ is a basic positive formula, $\mathfrak{A}_j \models \phi[\bar{b}, \bar{d}]$ and yet there is no $\bar{c}$ from $\mathfrak{A}_i$ for which both $\mathfrak{A}_i \models \phi[\bar{a}, \bar{c}]$ and $\Pi_\beta^+(\mathfrak{A}_i, \bar{a}, \bar{c}) \supseteq \Pi_\beta^+(\mathfrak{A}_j, \bar{b}, \bar{d})$.

For each such $\sigma \in S_\beta$ we may define the formula $\theta_\sigma(\bar{x})$ to be $\forall \bar{y} \neg \phi(\bar{x}, \bar{y}) \vee \neg \psi_\sigma(\bar{x}, \bar{y}))$ where ψ_σ is a recursive Π_β^+ formula obtained using the induction hypothesis having the property that, for all $\bar{c}$, $\Pi_\beta^+(\mathfrak{A}_i, \bar{a}, \bar{c}) \supseteq \Pi_\beta^+(\mathfrak{A}_j, \bar{b}, \bar{d})$ iff $\mathfrak{A}_i \models \psi_\sigma[\bar{a}, \bar{c}]$. We may therefore take the desired formula $\psi(\bar{x})$ to be equivalent to

$$\bigwedge\!\!\!\bigwedge_{\beta<\gamma} \bigwedge\!\!\!\bigwedge_{\sigma \in S_\beta} \theta_\sigma(\bar{x}).$$

It remains to see why such a ψ may be taken to be a recursive Π_γ^+ formula. We note that, by our assumptions, the conditions $\mathfrak{A}_j \models \phi[\bar{b}, \bar{d}]$ and $\mathfrak{A}_i \models \phi[\bar{a}, \bar{c}]$ are, for basic positive ϕ, recursive and that the condition $\Pi_\beta^+(\mathfrak{A}_i, \bar{a}, \bar{c}) \not\supseteq \Pi_\beta^+(\mathfrak{A}_j, \bar{b}, \bar{d})$ is uniformly Π_β^0. Thus, for $\beta \geqslant 1$, the set S_β is Π_β^0 uniformly in β.

If we define the formula $\psi_\beta(\bar{x})$ to be $\bigwedge\!\!\!\bigwedge_{\sigma \in S_\beta} \theta_\sigma(\bar{x})$, then for $\beta \geqslant 1$ we may re-express ψ_β as a recursive conjunction $\bigwedge\!\!\!\bigwedge_\sigma (\chi_\sigma \rightarrow \theta_\sigma)$, where each χ_σ is a recursive Π_β^+ sentence obtained recursively from σ. Because of the form of θ_σ, ψ_β may therefore be re-expressed again as a $\Pi_{\beta+1}^+$ formula.

In the case where $\beta = 0$, since the condition $\Pi_0^+(\mathfrak{A}_i, \bar{a}, \bar{c}) \supseteq \Pi_0^+(\mathfrak{A}_j, \bar{b}, \bar{d})$ is invariably true, the set S_0 is recursive (and each $\psi_\sigma(\bar{x}, \bar{y})$ for $\sigma \in S_0$ is the truth value T). Thus the formula $\psi_0(\bar{x})$ is a recursive conjunction and so may be re-expressed as a recursive Π_1^+ formula.

We have now shown that for each $\beta < \gamma$ the formula $\psi_\beta(\bar{x})$ may uniformly in β be expressed as a recursive $\Pi_{\beta+1}^+$ formula and thence as a recursive Π_γ^+ formula, from which it follows that the desired recursive conjunction $\bigwedge\!\!\!\bigwedge_{\beta<\gamma} \psi_\beta(\bar{x})$ may also be re-expressed as a recursive Π_γ^+ formula. □

Comments (on the assumptions). We note that the assumption that the positive open diagrams of the $\mathfrak{A}_i$ are recursive rather than r.e. is needed in the proof to show that the set S_1 is Π_1^0 rather than Δ_2^0, which is only critical in the case where $\gamma = 2$. The stronger assumption that the positive *existential* diagrams of the $\mathfrak{A}_i$ are recursive is needed to show that the set S_0 is recursive rather than Π_1^0, which is important only in the case where $\gamma = 1$. Without these assumptions, however, the induction appears to fail, at least for finite values of γ.

The assumptions of Lemma 4 imply, of course, that each of the structures $\mathfrak{A}_i$ is in fact a *recursive structure*. This does not mean that in this case the study of r.e. structures has become reduced to that of recursive structures. We may still, as in Section 6, construct isomorphic r.e. copies which are not necessarily recursive.

The need for such assumptions, here as elsewhere, seems to reflect the fact that for the recursive properties of r.e. or recursive copies of a structure to correlate neatly with the recursive syntactical properties of the structure, there must be at least one copy of the structure in which many additional features are recursive. A similar remark seems to apply for classes of structures.

As an immediate consequence of Lemma 4 we have:

Lemma 5. *Under the assumptions of Lemma 4, for each $\gamma \leqslant \alpha$, if $\Pi_\gamma^+(\mathfrak{A}_i) \nsubseteq \Pi_\gamma^+(\mathfrak{A}_j)$ then there is a* recursive *Π_γ^+ sentence true in $\mathfrak{A}_i$ but not in $\mathfrak{A}_j$.*

Proof. We may apply the conclusion of Lemma 4 to the empty sequence in $\mathfrak{A}_i$. □

6. Pairs of r.e. structures

Let $\mathfrak{A}$ and $\mathfrak{B}$ be r.e. structures. We write

$$\begin{cases} \mathfrak{A} & \text{if } \Pi_\alpha^+, \\ \mathfrak{B} & \text{if not} \end{cases}$$

to abbreviate the following statement: "For every Π_α^+ set, S, there is a recursive sequence $\langle \mathfrak{C}_n \rangle_{n \in \mathbb{N}}$ of r.e. structures such that, for each n,

$$\mathfrak{C}_n \cong \begin{cases} \mathfrak{A} & \text{if } n \in S, \\ \mathfrak{B} & \text{if } n \notin S." \end{cases}$$

Conditions for this property of $\mathfrak{A}$, $\mathfrak{B}$ and α, in the case of recursive structures $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}_n$, were discussed in [5] and several examples and generalizations were established. We show here that an analogous treatment is possible for r.e. structures, using Proposition 1.

Theorem. *Suppose that $\{\mathfrak{A}, \mathfrak{B}\}$ is an α-friendly pair of r.e. structures, where $\alpha < \omega_1^{\mathrm{CK}}$. Suppose also that the positive existential diagrams of both $\mathfrak{A}$ and $\mathfrak{B}$ are recursive. Then a necessary and sufficient condition for*

$$\begin{cases} \mathfrak{A} & \textit{if } \Pi_\alpha^+, \\ \mathfrak{B} & \textit{if not} \end{cases}$$

is that $\Pi_\alpha^+(\mathfrak{A}) \supseteq \Pi_\alpha^+(\mathfrak{B})$.

Proof. To see that the condition is necessary, we note that the assumptions of the theorem ensure the assumptions of Lemma 4. Thus, if $\Pi_\alpha^+(\mathfrak{A}) \nsupseteq \Pi_\alpha^+(\mathfrak{B})$ then, by Lemma 5, there is a *recursive* Π_α^+ sentence ψ which is true in $\mathfrak{B}$ but not in $\mathfrak{A}$. Let S be a Π_α^+ set which is not Σ_α^0 and suppose, for a contradiction, that there is a recursive sequence $\langle \mathfrak{C}_n \rangle_{n\in\mathbb{N}}$ of r.e. structures for which

$$\mathfrak{C}_n \cong \begin{cases} \mathfrak{A} & \text{if } n \in S, \\ \mathfrak{B} & \text{if } n \notin S. \end{cases}$$

Then we have $n \notin S$ iff $\mathfrak{C}_n \models \psi$, which is a Π^+_α property of n, contradicting the assumption that S is not Σ^0_α.

We proceed to show that the condition is *sufficient*, for which we need only the assumption that $\{\mathfrak{A}, \mathfrak{B}\}$ is α-friendly. Let us re-write $\mathfrak{A}$ and $\mathfrak{B}$ as $\mathfrak{A}_0$ and $\mathfrak{A}_1$, suppose that $\Pi^+_\alpha(\mathfrak{A}_0) \supseteq \Pi^+_\alpha(\mathfrak{A}_1)$ and let S be any Π^+_α set.

We construct, uniformly in n, suitable α-systems $\mathcal{S}_n$ and Δ^0_α inputs (u_n, l_n, p_n). In this case u_n, l_n and $\mathcal{S}_n = \mathcal{S} = (U, L, P, E, \langle \subseteq_\xi \rangle_{\xi<\alpha})$ are independent of n.

Let C be a fixed infinite recursive set, which will be the domain of each $\mathfrak{C}_n$. Let $U = \{0, 1\}$. Let L be the set of all triples (r, f, D) for which $r = 0$ or 1, f is a finite one–one partial function from C to $\mathfrak{A}_r$ and D is (the set of Gödel numbers of) a finite set of atomic sentences $\phi(\bar{c})$ of $L(C)$ for which $\bar{c} \in \mathrm{dom}(f)$ and $\mathfrak{A}_r \models \phi[f(\bar{c})]$.

We let P be the set of all finite sequences $(u_0, l_0, u_1, l_1, \ldots)$ where each $l_i = (r_i, f_i, D_i)$ such that, for each i:

(i) $r_i = u_i$.

(ii) $\mathrm{Dom}(f_i)$ contains the first i elements of C.

(iii) $\mathrm{Ran}(f_i)$ contains the first i elements of $\mathfrak{A}_{r_i}$.

(iv) D_i contains each atomic sentence $\phi(\bar{c})$ of $L(C)$ for which $\phi(f_i(\bar{c}))$ has appeared in the first i steps of the enumeration of the diagram of $\mathfrak{A}_{r_i}$.

(v) If $u_i = 1$, then $u_{i+1} = 1$.

(vi) If $u_{i+1} = u_i$, then $f_{i+1} \supseteq f_i$.

If $l = (r, f, D)$, then we define $E(l) = D$. If also $l' = (r', f', D')$, then we define $l \subseteq_\xi l'$ iff $D \subseteq D'$ and $f \lhd^+_\xi f'$.

As required in the definition of an α-system, we note that the set L is r.e., since $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are r.e. structures, and that the relations $\subseteq_\xi$ on L are r.e. uniformly in $\xi < \alpha$ by the assumption that the family $\{\mathfrak{A}_0, \mathfrak{A}_i\}$ is α-friendly.

Since the set S is Π^0_α, there is a $\{0, 1\}$-valued Δ^0_α function $h(n, i)$ such that, for each n, $h(n, i)$ is non-decreasing and $n \in S$ iff $h(n, i) = 0$ for all i. So, uniformly in n, we may define an input (u, l, p_n) for $\mathcal{S}$ where $u = 0$, $l = (0, \emptyset, \emptyset)$ and $p_n(u_0, l_0, \ldots, u_i, l_i) = h(n, i)$ for each $i \geqslant 0$.

A conservative Δ^0_α run $u_0, l_0, u_1, l_1, \ldots$ of the input (u, l, p_n) on $\mathcal{S}$ determines, therefore, a chain of finite sets D_i of atomic sentences of $L(C)$ whose r.e. union is the diagram of the desired r.e. structure $\mathfrak{C}_n$ and a sequence f_i of finite partial functions forming a chain (from some point on, in the case where $n \notin S$) whose union is an isomorphism from $\mathfrak{C}$ either to $\mathfrak{A}_0$ if $n \in S$ or to $\mathfrak{A}_1$ if $n \notin S$. The uniformity of the enumeration of the $\mathfrak{C}_n$ follows from the uniformity of the p_n and the second part of Proposition 1.

It remains only to verify that the system $\mathcal{S}$ we have defined is in fact an

α-system. Conditions (1) and (2) follow immediately from the definitions and from Lemma 1, while condition (3) follows from the definitions alone.

To establish condition (4), suppose that $\alpha > \xi_1 > \xi_2 > \cdots > \xi_k$, that $(u_0, l_0, \ldots, u_i, l_i, u_{i+1}) \in P$ and that $l_i = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \cdots \subseteq_{\xi_{k-1}} m_k$.

Let $m_j = (r_j, f_j, D_j)$. Then we have $f_1 \lhd^+_{\xi_1} f_2 \lhd^+_{\xi_2} \cdots \lhd^+_{\xi_{k-1}} f_k$ and $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k$. Let $\phi(\bar{c})$ be the conjunction of the sentences of D_k. Then, since $m_k \in L$, ϕ is true for $f_k(\bar{c})$ in $\mathfrak{A}_{r_k}$ and so, by Lemma 3, there exists $g \supseteq f_1$ such that ϕ is true for $g(\bar{c})$ in $\mathfrak{A}_{u_i}$ and $f_j \lhd_{\xi_j} g$ for each $j = 1, 2, \ldots, k$.

In the case where $u_{i+1} = u_i$, we may take the l_{i+1} required by condition (4) to be (u_{i+1}, f, D) where $f \supseteq g$ and $D \supseteq D_k$ are chosen to satisfy the definitions of L and P.

In the remaining case, where $u_i = 0$ and $u_{i+1} = 1$, then g is a finite partial one–one function from C to $\mathfrak{A}_0$ whose domain, without loss of generality, is $\bar{c}$ and such that ϕ is true in $\mathfrak{A}_0$ for $g(\bar{c})$. Now since $\xi_1 < \alpha$ and since we are assuming that $\Pi^+_\alpha(\mathfrak{A}_0) \supseteq \Pi^+_\alpha(\mathfrak{A}_1)$, we may apply Lemma 1 and the definition of $\leqslant^+_\alpha$ to the empty sequences from $\mathfrak{A}_0$ and $\mathfrak{A}_1$ and deduce that there exists $\bar{b}$ from $\mathfrak{A}_1$ for which $g(\bar{c}) \leqslant^+_{\xi_1} \bar{b}$ and ϕ is true for $\bar{b}$ in $\mathfrak{A}_1$. We may therefore define g' by $g'(\bar{c}) = \bar{b}$, so that $g \lhd^+_{\xi_1} g'$ and take l_{i+1} in this case to be $(1, f, D)$ where $f \supseteq g'$ and $D \supseteq D_k$ are chosen to satisfy the definitions of L and P. $\square$

Comment 1. Unlike the corresponding argument for Theorem 3.1 of [5], we have not needed here to give separate arguments in the two cases where α is a successor ordinal and α is a limit ordinal.

Comment 2. A virtually identical proof of Theorem 3.1 of [5] itself, in which all the structures considered are required to be recursive rather than r.e., can be given, using our new α-systems. The changes needed are merely that the sets D in the definition of L consist of finite sets of atomic *and negated atomic* formulas and the relations $\subseteq_\xi$ are defined in terms of the relations $\lhd_\xi$ between functions instead of the $\lhd^+_\xi$. (The relations $\lhd_\xi$ and $\leqslant_\xi$ were paraphrased in [5] but were treated in [2] and [3] and have the same connections with Π_ξ formulas as do the present $\lhd^+_\xi$ and $\leqslant^+_\xi$ with Π^+_ξ formulas.)

Comment 3. On the other hand, our present result already yields Theorem 3.1 of [5] directly in the following way. As remarked in Section 1, each structure $\mathfrak{A} = (A, \langle R_i \rangle_{i \in I})$ for L yields an expansion $\mathfrak{A}' = (A, \langle R_i \rangle_{i \in I}, \langle \bar{R}_i \rangle_{i \in I})$ for L' such that $\mathfrak{A}$ is recursive iff $\mathfrak{A}'$ is r.e.

Each Π_ξ formula ψ of L gives rise to an equivalent Π^+_ξ formula ψ' of L' by replacing appropriate occurrences of the symbols $\neg P_i$ and P_i by the new symbols $\bar{P}_i$ and $\neg \bar{P}_i$ respectively. This transformation can be reversed and is entirely effective, so that *recursive* Π_ξ formulas of L correspond to *recursive* Π^+_ξ formulas of L'. Moreover, the finitary quantifier-free and existential formulas of L can be transformed into equivalent finitary positive and positive existential formulas of L' respectively, and vice versa.

It follows that if structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the assumptions of Theorem 3.1 of [5], namely that $\{\mathfrak{A}, \mathfrak{B}\}$ is an α-friendly family of recursive structures having recursive existential diagrams, then $\{\mathfrak{A}', \mathfrak{B}'\}$ is an α-friendly family of *r.e.* structures having recursive *positive* existential diagrams, so $\mathfrak{A}'$ and $\mathfrak{B}'$ satisfy the assumptions of the present theorem. The condition $\Pi_\alpha(\mathfrak{A}) \supseteq \Pi_\alpha(\mathfrak{B})$ is likewise equivalent to the condition $\Pi^+_\alpha(\mathfrak{A}') \supseteq \Pi^+_\alpha(\mathfrak{B}')$. If $\langle \mathfrak{C}_n \rangle_{n\in\mathbb{N}}$ is a recursive sequence of recursive structures for which

$$\mathfrak{C}_n \cong \begin{cases} \mathfrak{A} & \text{if } n \in S, \\ \mathfrak{B} & \text{if } n \notin S \end{cases}$$

then the corresponding expansion $\mathfrak{C}'_n$ form a recursive sequence of r.e. structures for which

$$\mathfrak{C}'_n \cong \begin{cases} \mathfrak{A}' & \text{if } n \in S, \\ \mathfrak{B}' & \text{if } n \neq S, \end{cases}$$

while if $\langle \mathfrak{C}'_n \rangle_{n\in\mathbb{N}}$ is a recursive sequence of r.e. structures satisfying this second condition then, by the nature of $\mathfrak{A}'$ and $\mathfrak{B}'$, each $\mathfrak{C}'_n$ is in fact a recursive structure and the corresponding reducts $\mathfrak{C}_n$ form a recursive sequence of recursive structures satisfying the first condition.

Thus, applying our present theorem to the family $\{\mathfrak{A}', \mathfrak{B}'\}$ yields Theorem 3.1 of [5] for the family $\{\mathfrak{A}, \mathfrak{B}\}$.

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